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A_ν-OPERATOR ON COMPLETE FOLIATED RIEMANNIAN MANIFOLDS

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ABSTRACT

We give a generalization of the result obtained by C. Currás-Bosch. We consider the A_ν-operator associated to a transverse Killing field ν on a complete foliated Riemannian manifold $(M, \mathcal{F}, g)$. Under a certain assumption, we prove that, for each $x \in M$, $(A_\nu)_x$ belongs to the Lie algebra of the linear holonomy group $\Psi_\nabla(x)$. A special case of our result, the version of the foliation by points, implies the results given by B. Kostant (compact case) and C. Currás-Bosch (non-compact case).

1. Introduction

The following Kostant's result is well-known: If X is a Killing vector field on a compact Riemannian manifold M, then, for each $x \in M$, $(A_X)_x$ belongs to the Lie algebra of the linear holonomy group $\Psi(x)$ ([6], p. 247).

The purpose of this note is that of extending the above result to the case of complete foliated Riemannian manifold. Our result is

THEOREM. *Let $(M, \mathcal{F}, g)$ be a connected, orientable, complete, foliated Riemannian manifold with a minimal foliation $\mathcal{F}$ and a bundle-like metric g with respect to $\mathcal{F}$. Let ν be a transverse Killing field with finite global norm. Then, for each $x \in M$, $(A_\nu)_x$ belongs to the Lie algebra of the linear holonomy group $\Psi_\nabla(x)$, where ∇ is the transversal Riemannian connection of $\mathcal{F}$.*

If $\mathcal{F}$ is the foliation by points, then ν is a Killing vector field on M with finite global norm and $\Psi_\nabla(x) = \Psi(x)$. Thus we have the result given by C. Currás-Bosch [2]. If M is compact and $\mathcal{F}$ is the foliation by points, then we have the above Kostant's result.

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REMARK. Let $(M, \mathscr{F}, g)$ be as in Theorem. If the Ricci operator of $\mathscr{F}$ is non-positive everywhere and negative for at least one point of M, then every transverse Killing field with finite global norm is trivial ([12]).

We shall be in C^∞-category and deal only with connected and orientable manifolds (without boundary). The author wishes to express his thanks to the referee for kind suggestions.

2. Tansverse Killing fields

Let $(M, \mathscr{F}, g)$ be a complete foliated Riemannian manifold of dimension $p+q$ with a foliation $\mathscr{F}$ of codimension q and a bundle-like metric g in the sense of B. L. Reinhart [7]. The foliation $\mathscr{F}$ is given by an integrable subbundle E of the tangent bundle TM over M. The quotient bundle $Q := TM/E$ is called the normal bundle of $\mathscr{F}$. Let π: $TM \to Q$ be the natural projection. The bundle-like metric g defines a map σ: $Q \to TM$ with $\pi \circ \sigma =$ identity and induces a metric g_Q in Q ([3], [4]). There exist local orthonormal adapted frames $\{E_i, E_\alpha\}$ to $\mathscr{F}$ ([8], [11]). Here and subsequently, we use the following convention on the range of indices: $1 \leqq A, B \leqq p+q$, $1 \leqq i, j \leqq p$, and $p+1 \leqq \alpha, \beta \leqq p+q$.

Let ∇ be the *transversal Riemannian connection* in Q ([3], [4]). We have that $i(X)R_\nabla = 0$ for all $X \in \Gamma(E)$, where $i(X)$ denotes the interior product with respect to X, and R_∇ denotes the curvature of ∇ ([3]).

Let $V(\mathscr{F})$ be the space of all vector fields X on M satisfying $[X, Z] \in \Gamma(E)$ for all $Z \in \Gamma(E)$. We define $\theta(X)$: $\Gamma(Q) \to \Gamma(Q)$ for $X \in V(\mathscr{F})$ by $\theta(X)\nu := \pi([X, Y])$ for all $\nu \in \Gamma(Q)$ and $Y \in \Gamma(TM)$ with $\pi(Y) = \nu$. Then we have

DEFINITION 1 ([4]). If $X \in V(\mathscr{F})$ satisfies $\theta(X)g_Q = 0$, then $\pi(X) \in \Gamma(Q)$ is called a *transverse Killing field* of $\mathscr{F}$.

DEFINITION 2 ([4]). The operator A_ν: $\Gamma(Q) \to \Gamma(Q)$ for $\nu \in \Gamma(Q)$ is defined by $A_\nu(\mu) := -\nabla_Y \nu$, where $Y \in \Gamma(TM)$ with $\pi(Y) = \mu$.

PROPOSITION 3 ([4]). *If $\nu = \pi(X)$ is a transverse Killing field of $\mathscr{F}$, then*
(i) $g_Q(A_\nu(\mu), \tau) + g_Q(\mu, A_\nu(\tau)) = 0$ *for* $\mu, \tau \in \Gamma(Q)$,
(ii) $\nabla_Y A_\nu = R_\nabla(\nu, \mu)$ *for* $Y \in \Gamma(TM)$ *with* $\pi(Y) = \mu$.

Let $\Gamma_0(Q)$ be the space of all sections of Q with compact support in M. We define the global scalar product $\langle\!\langle\ ,\ \rangle\!\rangle$ by $\langle\!\langle \nu, \mu \rangle\!\rangle := \int_M g_Q(\nu, \mu)\, dV$ for all $\nu, \mu \in \Gamma_0(Q)$, where dV denotes the volume element of M. Let $L_2(M, Q)$ be the completion of $\Gamma_0(Q)$ with respect to $\langle\!\langle\ ,\ \rangle\!\rangle$. We set $\|\nu\|^2 := \langle\!\langle \nu, \nu \rangle\!\rangle$.

DEFINITION 4 ([10], [12]). A transverse field ν has *finite global norm* if $\nu \in L_2(M, Q) \cap \Gamma(Q)$.

Let x_0 be a point of M and fix it. For each $x \in M$, we denote by $\rho(x)$ the geodesic distance from x_0 to x. For any $r > 0$, we set $B(r) := \{x \in M \mid \rho(x) < r\}$. Then there exists a family $\{w_r\}_{r>0}$ of cut-off functions on M ([1], [5], [10], [12]). We note that, for $\nu \in L_2(M, Q) \cap \Gamma(Q)$, $w_r\nu$ lies in $\Gamma_0(Q)$ and $w_r\nu \to \nu$ $(r \to \infty)$ in the strong sense.

The exterior derivative d has the decomposition: $d = d' + d'' + d'''$ ([5], [9]). Then we have $dw_r = d'w_r + d''w_r$ and $d''w_r = \sum_{\alpha=p+1}^{p+q} E_\alpha(w_r)E_\alpha^*$, where $\{E_A^*\}$ denotes the dual frame to $\{E_A\}$. For $\nu \in \Gamma(Q)$, we may regard $d''w_r \otimes \nu$ as a linear map: $\Gamma(Q) \to \Gamma(Q)$ with $d''w_r \otimes \nu(\mu) := d''w_r(\sigma(\mu)) \cdot \nu$.

LEMMA 5 ([1], [5], [10], [12]). *For any $\nu \in \Gamma(Q)$, there exists a positive constant C^* independent of r such that*

$$\| d''w_r \otimes \nu \|_{B(2r)}^2 \leqq C^* r^{-2} \| \nu \|_{B(2r)}^2$$

where

$$\| \cdot \|_{B(2r)}^2 = \langle\!\langle \cdot , \cdot \rangle\!\rangle_{B(2r)} = \int_{B(2r)} g_Q(\cdot , \cdot) dV.$$

3. Proof of Theorem

Let $(M, \mathcal{F}, g)$ be as in Theorem. We remark that a leaf L of $\mathcal{F}$ is minimal if $\pi(\sum_{i=1}^{p} \nabla_{E_i}^M E_i)_x = 0$ at each $x \in L$, where ∇^M denotes the Levi–Civita connection with respect to g, and $\mathcal{F}$ is *minimal* if all the leaves of $\mathcal{F}$ are minimal ([3], [4], [8], [12]). We define an operator div_∇: $\Gamma(Q) \to \mathbf{R}$ by

$$\mathrm{div}_\nabla \nu := \sum_{\alpha=p+1}^{p+q} g_Q(\nabla_{E_\alpha}\nu, \pi(E_\alpha))$$

for all $\nu \in \Gamma(Q)$. This is independent of the choice of the local adapted frames. Let I: $\Gamma(Q) \to \Gamma(Q)$ be the identity map. We first have the following proposition.

PROPOSITION 6. *For $\nu \in \Gamma(Q)$,*

$$\int_M w_r^2 \mathrm{div}_\nabla \nu dV + \langle\!\langle 2d''w_r \otimes \nu, w_r I \rangle\!\rangle_{B(2r)} = 0.$$

PROOF. We have

$$
\begin{aligned}
\operatorname{div}(\sigma(w_r^2\nu)) &= \operatorname{div}(w_r^2\sigma(\nu)) \\
&= \sum_{\alpha=p+1}^{p+q} g(2w_rE_\alpha(w_r)\sigma(\nu), E_\alpha) + \sum_{i=1}^{p} g(w_r^2\nabla^M_{E_i}\sigma(\nu), E_i) \\
&\quad + \sum_{\alpha=p+1}^{p+q} g(w_r^2\nabla^M_{E_\alpha}\sigma(\nu), E_\alpha) \\
&= \sum_{\alpha=p+1}^{p+q} g(2w_rd''w_r(E_\alpha)\sigma(\nu), E_\alpha) - \sum_{i=1}^{p} g(w_r^2\sigma(\nu), \nabla^M_{E_i}E_i) \\
&\quad + \sum_{\alpha=p+1}^{p+q} g_Q(w_r^2\pi(\nabla^M_{E_\alpha}\sigma(\nu)), \pi(E_\alpha)) \\
&= \sum_{\alpha=p+1}^{p+q} g_Q(2w_rd''w_r(E_\alpha)\pi(\sigma(\nu)), \pi(E_\alpha)) \\
&\quad + \sum_{\alpha=p+1}^{p+q} g_Q(w_r^2\nabla_{E_\alpha}\nu, \pi(E_\alpha)) \qquad \text{(by the minimality of } \mathcal{F}) \\
&= \sum_{\alpha=p+1}^{p+q} g_Q(2d''w_r(E_\alpha)\nu, w_r\pi(E_\alpha)) + w_r^2 \operatorname{div}_\nabla \nu.
\end{aligned}
$$

As $w_r^2\sigma(\nu)$ has compact support contained in $B(2r)$, by Green's theorem, we complete the proof of Proposition 6. ∎

Corollary 7. *If M in Theorem is compact, then*

$$\int_M \operatorname{div}_\nabla \nu dV = 0$$

for $\nu \in \Gamma(Q)$.

Corollary 8. *If M is as in Theorem and ν is a transverse Killing field, then it holds that* $\operatorname{div}_\nabla \nu = 0$.

Now, let $p: L(Q) \to M$ be the linear frame bundle of Q with the structure group $O(q)$. Let $\Psi_\nabla(x)$ be the *linear holonomy group* (with reference point x) of the connection form on $L(Q)$ associated to ∇ ([6, Chapters II and III]). We denote by $\mathfrak{G}_\nabla(x)$ the Lie algebra of the linear holonomy group $\Psi_\nabla(x)$ for each $x \in M$. Let $\mathscr{E}(x)$ be the Lie algebra of skew-symmetric endomorphisms of Q_x, and let $\mathfrak{G}_\nabla^\perp(x)$ be the orthogonal complement of $\mathfrak{G}_\nabla(x)$ in $\mathscr{E}(x)$ with respect to the inner product induced from g_Q. For a transverse Killing field ν, we set

$$A_\nu = S_\nu + B_\nu$$

where $(S_\nu)_x \in \mathfrak{G}_\nabla(x)$ and $(B_\nu)_x \in \mathfrak{G}_\nabla^\perp(x)$ for each $x \in M$. In the same way as for

Lemma in [6, p. 247], we have that $\nabla_X B_\nu = 0$ for all $X \in \Gamma(TM)$. If we prove $B_\nu = 0$ for any transverse Killing field ν with finite global norm, then we have the proof of theorem. We show that $B_\nu = 0$ in the same way as [2].

Let ν be a transverse Killing field with finite global norm. Since B_ν is skew-symmetric and $\operatorname{div}_\nabla B_\nu(\nu) = -\langle B_\nu, B_\nu \rangle$, we have

$$\int_M w_r^2 \operatorname{div}_\nabla B_\nu(\nu) dV = -\| w_r B_\nu \|^2_{B(2r)},$$

and

$$\langle\!\langle 2d'' w_r \otimes B_\nu(\nu), w_r I \rangle\!\rangle_{B(2r)} = -\langle\!\langle 2d'' w_r \otimes \nu, w_r B_\nu \rangle\!\rangle_{B(2r)}.$$

By Proposition 6, we have

$$\| w_r B_\nu \|^2_{B(2r)} + \langle\!\langle 2d'' w_r \otimes \nu, w_r B_\nu \rangle\!\rangle_{B(2r)} = 0.$$

By Schwarz inequality and Lemma 5, we have

$$\begin{aligned} |\langle\!\langle d'' w_r \otimes \nu, w_r B_\nu \rangle\!\rangle_{B(2r)}| &\leqq \| 2d'' w_r \otimes \nu \|_{B(2r)} \| w_r B_\nu \|_{B(2r)} \\ &\leqq 2^{-1} \| w_r B_\nu \|^2_{B(2r)} + 2C^* r^{-2} \| \nu \|^2_{B(2r)}. \end{aligned}$$

Thus we have

$$\| w_r B_\nu \|^2_{B(2r)} \leqq 4C^* r^{-2} \| \nu \|^2_{B(2r)}.$$

Since ν has finite global norm, letting $r \to \infty$, we have

$$\lim_{r \to \infty} \| w_r B_\nu \|^2_{B(2r)} \leqq 0.$$

Therefore, we have that $B_\nu = 0$.

By examples in [2] and [8], we can construct examples of M and transverse Killing fields ν with infinite global norms on M such that A_ν does not belong to the Lie algebra of the linear holonomy group $\Psi_\nabla(x)$ for each $x \in M$.

REFERENCES

1. A. Andreotti and E. Vesentini, *Carleman estimates for the Laplace–Beltrami equation on complex manifolds*, Inst. Hautes Etudes Sci. Publ. Math. **25** (1965), 313–362.

2. C. Currás-Bosch, *A_X-operator on complete Riemannian manifolds*, Israel J. Math. **53** (1986), 315–320.

3. F. W. Kamber and Ph. Tondeur, *Harmonic foliations*, Lecture Notes in Math. **949**, Springer-Verlag, Berlin, Heidelberg and New York, 1982, pp. 87–121.

4. F. W. Kamber and Ph. Tondeur, *Infinitesimal automorphisms and second variation of the energy for harmonic foliations*, Tohoku Math. J. **34** (1982), 525–538.

5. H. Kitahara, *Nonexistence of nontrivial* □″*-harmonic* 1*-forms on a complete foliated riemannian manifold*, Trans. Amer. Math. Soc. **262** (1980), 429–435.

6. S. Kobayashi and K. Nomizu, *Foundations of Differential Geometry*, Vol. I, Interscience, New York, 1963.

7. B. L. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. of Math. **69** (1959), 119–132.

8. R. Takagi and S. Yorozu, *Minimal foliations on Lie groups*, Tohoku Math. J. **36** (1984), 541–554.

9. I. Vaisman, *Cohomology and Differentiable Forms*, Marcel Dekker, New York, 1973.

10. S. Yorozu, *Killing vector fields on complete Riemannian manifolds*, Proc. Amer. Math. Soc. **84** (1982), 115–120.

11. S. Yorozu, *Behavior of geodesics in foliated manifolds with bundle-like metrics*, J. Math. Soc. Japan **35** (1983), 251–272.

12. S. Yorozu, *The non-existence of Killing fields*, Tohoku Math. J. **36** (1984), 99–105.